Oscillation and nonoscillation theorems for a class of fourth order differential equations with deviating arguments (Dynamics of Functional Equations and Related Topics)

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数理解析研究所講究録 1254 巻 2002 年 193-201

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0. Introduction

We consider the oscillatory and nonoscillatory behavior of fourth order nonlinear functional differential equations of the type

$$(|y''(t)|^{\alpha}\mathrm{sgn}\, y''(t))'' + q(t)|y(g(t))|^{\beta}\mathrm{sgn}\, y(g(t)) = 0 \tag{A}$$

for which the following conditions are always assumed to hold:

(a) α and β are positive constants;

(b) $q : [0,\infty) \to (0,\infty)$ is a continuous function;

(c) $g : [0,\infty) \to (0,\infty)$ is a continuously differentiable function such that $g'(t) > 0$, $t \geq 0$, and $\lim_{t\to\infty} g(t) = \infty$.

By a solution of (A) we mean a function $y : [T_y, \infty) \to \mathbb{R}$ which is twice continuously differentiable together with $|y''|^{\alpha}\mathrm{sgn}\, y''$ and satisfies the equation (A) at all sufficiently large t. Those solutions which vanish in a neighborhood of infinity will be excluded from our consideration. A solution of (A) is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory. This means that a solution $y(t)$ is oscillatory if and only if there is a sequence $\{t_i\}_{i=1}^{\infty}$ such that $t_i \to \infty$ and $y(t_i) = 0$ $(i = 1, 2, \cdots)$, and a solution $y(t)$ is nonoscillatory if and only if $y(t) \neq 0$ for all large t.

In Section 1 we study the problem of existence of nonoscillatory solutions of (A). The set of all nonoscillatory solutions of (A) is classified into six disjoint classes according to their asymptotic behavior at ∞, and criteria are established for the existence of solutions belonging to each of these six classes. Some of the criteria are shown to be sharp enough.

In Section 2 we next attempt to derive criteria for the oscillation of all solutions of (A). Our derivation depends heavily on oscillation theory of fourth order nonlinear ordinary differential equations

$$(|y''|^{\alpha}\mathrm{sgn}\, y'')'' + q(t)|y|^{\beta}\mathrm{sgn}\, y = 0 \tag{B}$$

recently developed by Wu [6], in conjunction with a comparison principle which enables us to deduce oscillation of an equation of the form (A) from that of a similar equation with a different functional argument. As a result, we are able to demonstrate the existence of classes of equations of the form (A) for which sharp oscillation criteria can be established.

We note that oscillation properties of second order functional differential equations involving nonlinear Sturm-Liouville type differential operators have been investigated by Kusano and Lalli [1], Kusano and Wang [3] and Wang [5]. The present paper is a first step toward generalizing the above results to higher order functional differential equations whose principal parts are composed of genuinely nonlinear differential operators.

1. Nonoscillation theorems

The purpose of this section is to make a detailed analysis of the structure of the set of all possible nonosciillatory solutions of the equation (A), which can also be written as

$$\text{(A)} \qquad ((y''(t))^{\alpha*})'' + q(t)(y(g(t)))^{\beta*} = 0$$

in terms of the asterisk notation

$$\text{(1.1)} \qquad \xi^{\gamma*} = |\xi|^{\gamma}\operatorname{sgn}\xi = |\xi|^{\gamma-1}\xi, \qquad \xi \in \mathbb{R}, \quad \gamma > 0.$$

A) *Classification of nonoscillatory solutions.* It suffices to restrict our consideration to eventually positive solutions of (A), since if $y(t)$ is a solution of (A) then so is $-y(t)$. Let $y(t)$ be one such solution. Then, as is easily verified, $y(t)$ satisfies either

I: $y'(t) > 0, \quad y''(t) > 0, \quad ((y''(t))^{\alpha*})' > 0 \qquad$ for all large t

or

II: $y'(t) > 0, \quad y''(t) < 0, \quad ((y''(t))^{\alpha*})' > 0 \qquad$ for all large t.

(See Wu [6].) It follows that $y(t)$, $y'(t)$, $y''(t)$ and $((y''(t))^{\alpha*})'$ are eventually monotone, so that they tend to finite or infinite limits as $t \to \infty$. Let

$$\lim_{t\to\infty} y^{(i)}(t) = \omega_i, \quad i = 0,\ 1, 2, \qquad \text{and} \qquad \lim_{t\to\infty}((y''(t))^{\alpha*})' = \omega_3.$$

It is clear that ω_3 is a finite nonnegative number. One can easily shows that:

(i) if $y(t)$ satisfies I, then the set of its asymptotic values $\{\omega_i\}$ falls into one of the following three cases:

I_1: $\quad \omega_0 = \omega_1 = \omega_2 = \infty, \quad \omega_3 \in (0,\infty);$

I_2: $\quad \omega_0 = \omega_1 = \omega_2 = \infty, \quad \omega_3 = 0;$

I_3: $\quad \omega_0 = \omega_1 = \infty, \quad \omega_2 \in (0,\infty), \quad \omega_3 = 0.$

(ii) if $y(t)$ satisfies II, then the set of its asymptotic values $\{\omega_i\}$ falls into one of the following three cases:

II_1: $\quad \omega_0 = \infty, \quad \omega_1 \in (0,\infty), \quad \omega_2 = \omega_3 = 0;$

II_2: $\quad \omega_0 = \infty, \quad \omega_1 = \omega_2 = \omega_3 = 0;$

II_3: $\quad \omega_0 \in (0,\infty), \quad \omega_1 = \omega_2 = \omega_3 = 0.$

Equivalent expressions for these six classes of positive solutions of (A) are as follows:

I_1: $\quad \displaystyle\lim_{t\to\infty} \frac{y(t)}{t^{2+\frac{1}{\alpha}}} = \text{const} > 0;$

I_2: $\quad \displaystyle\lim_{t\to\infty} \frac{y(t)}{t^{2+\frac{1}{\alpha}}} = 0, \quad \lim_{t\to\infty} \frac{y(t)}{t^2} = \infty;$

I$_3$: $\lim_{t\to\infty} \frac{y(t)}{t^2} = \text{const} > 0;$

II$_1$: $\lim_{t\to\infty} \frac{y(t)}{t} = \text{const} > 0;$

II$_2$: $\lim_{t\to\infty} \frac{y(t)}{t} = 0, \quad \lim_{t\to\infty} y(t) = \infty;$

II$_3$: $\lim_{t\to\infty} y(t) = \text{const} > 0.$

B) *Integral representations for nonoscillatory solutions.* We want to establish the existence of positive solutions of (A) belonging to each of the above six classes. For this purpose a crucial role will be played by integral representations for those six types of solutions of (A) as derived below.

Let $y(t)$ be a positive solution of (A) such that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq T > 0$. Integrating (A) from t to ∞ gives

$$(1.2) \qquad ((y''(t))^{\alpha *})' = \omega_3 + \int_t^\infty q(s)(y(g(s)))^\beta ds, \quad t \geq T.$$

We now integrate (1.2) three times over $[T,t]$ to obtain

$$(1.3) \quad y(t) = k_0 + k_1(t-T) + \int_T^t (t-s)\left[k_2^\alpha + \int_T^s \left(\omega_3 + \int_r^\infty q(\sigma)(y(g(\sigma)))^\beta d\sigma\right) dr\right]^{\frac{1}{\alpha}} ds,$$

for $t \geq T$, which is an integral representation for a solution $y(t)$ of type I$_1$, where $k_0 = y(T)$, $k_1 = y'(T)$ and $k_2 = y''(T)$ are nonnegative constants. A type-I$_2$ solution $y(t)$ of (A) is expressed by (1.3) with $\omega_3 = 0$.

If $y(t)$ is a solution of type I$_3$, then, first integrating (1.2) from t to ∞ and then integrating the resulting equation twice from T to t, we have

$$(1.4) \quad y(t) = k_0 + k_1(t-T) + \int_T^t (t-s)\left[\omega_2^\alpha - \int_s^\infty (r-s)q(r)(y(g(r)))^\beta dr\right]^{\frac{1}{\alpha}} ds, \quad t \geq T.$$

An integral representation for a solution $y(t)$ of type II$_1$ is derived by integrating (1.2) with $\omega_3 = 0$ twice from t to ∞ and then once from T to t:

$$(1.5) \qquad y(t) = k_0 + \int_T^t \left(\omega_1 + \int_s^\infty \left[\int_r^\infty (\sigma - r)q(\sigma)(y(g(\sigma)))^\beta d\sigma\right]^{\frac{1}{\alpha}} dr\right) ds, \qquad t \geq T.$$

An expression for a type-II$_2$ solution is given by (1.5) with $\omega_1 = 0$. If $y(t)$ is a solution of type II$_3$, then three integrations of (A) with $\omega_3 = 0$ yield

$$(1.6) \qquad y(t) = \omega_0 - \int_t^\infty (s-t)\left[\int_s^\infty (r-s)q(r)(y(g(r)))^\beta dr\right]^{\frac{1}{\alpha}} ds, \qquad t \geq T.$$

C) *Nonoscillation criteria (necessary and sufficient conditions).* The four types I$_1$, I$_3$, II$_1$ and II$_3$ of solutions are taken up and necessary and sufficient conditions are established for the existence of positive solutions of these four types for (A).

THEOREM 1.1. *The equation* (A) *has a positive solution of type* I_1 *if and only if*

$$\int_0^\infty (g(t))^{(2+\frac{1}{\alpha})\beta} q(t)dt < \infty. \tag{1.7}$$

THEOREM 1.2. *The equation* (A) *has a positive solution of type* I_3 *if and only if*

$$\int_0^\infty t(g(t))^{2\beta} q(t)dt < \infty. \tag{1.8}$$

THEOREM 1.3. *The equation* (A) *has a positive solution of type* II_1 *if and only if*

$$\int_0^\infty \left[\int_t^\infty (s-t)(g(s))^\beta q(s)ds\right]^{\frac{1}{\alpha}} dt < \infty. \tag{1.9}$$

THEOREM 1.4. *The equation* (A) *has a positive solution of type* II_3 *if and only if*

$$\int_0^\infty t\left[\int_t^\infty (s-t)q(s)ds\right]^{\frac{1}{\alpha}} dt < \infty. \tag{1.10}$$

THE PROOF OF THEOREM 1.1. Suppose that (A) has a solution $y(t)$ of type I_1. Then, it satisfies (1.3) for $t \geq T$, $T > 0$ being sufficiently large, which implies that

$$\int_T^\infty q(t)(y(g(t)))^\beta dt < \infty.$$

This, combined with the asymptotic relation $\lim_{t\to\infty} y(t)/t^{2+\frac{1}{\alpha}} = \text{const} > 0$, shows that the condition (1.7) is satisfied.

Now suppose that (1.7) holds. Let $k > 0$ be given arbitrarily constant and choose $T > 0$ large enough so that

$$\left(\frac{\alpha^2}{(\alpha+1)(2\alpha+1)}\right)^\beta \int_T^\infty (g(t))^{(2+\frac{1}{\alpha})\beta} q(t)dt \leq \frac{(2k)^\alpha - k^\alpha}{(2k)^\beta}. \tag{1.11}$$

Put $T_* = \min\{T, \inf_{t\geq T} g(t)\}$, and define

$$\begin{aligned} &G(t,T) = \int_T^t (t-s)(s-T)^{\frac{1}{\alpha}} ds = \frac{\alpha^2}{(\alpha+1)(2\alpha+1)}(t-T)^{2+\frac{1}{\alpha}}, \quad t \geq T, \\ &G(t,T) = 0, \quad t \leq T. \end{aligned} \tag{1.12}$$

Let $Y \subset C[T_*,\infty)$ and $\mathcal{F} : Y \to C[T_*,\infty)$ be defined as follows:

$$Y = \{y \in C[T_*,\infty) : \ kG(t,T) \leq y(t) \leq 2kG(t,T), \ t \geq T_*\}, \tag{1.13}$$

$$\begin{aligned} &\mathcal{F}y(t) = \int_T^t (t-s)\left[\int_T^s \left(k^\alpha + \int_r^\infty q(\sigma)(y(g(\sigma)))^\beta d\sigma\right) dr\right]^{\frac{1}{\alpha}} ds, \quad t \geq T \\ &\mathcal{F}y(t) = 0, \quad T_* \leq t \leq T. \end{aligned} \tag{1.14}$$

If $y \in Y$, then for $t \geq T$

$$\mathcal{F}y(t) \geq k \int_T^t (t-s)(s-T)^{\frac{1}{\alpha}} ds = kG(t,T)$$

and

$$\mathcal{F}y(t) \leq \int_T^t (t-s) \left[\int_T^s \left(k^\alpha + \int_r^\infty q(\sigma)(2kG(g(\sigma),T))^\beta d\sigma \right) dr \right]^{\frac{1}{\alpha}} ds$$

$$\leq \int_T^t (t-s) \left[\int_T^s \left(k^\alpha + \left(\frac{\alpha^2 \cdot 2k}{(\alpha+1)(2\alpha+1)} \right)^\beta \int_r^\infty q(\sigma)(g(\sigma))^{(2+\frac{1}{\alpha})\beta} d\sigma \right) dr \right]^{\frac{1}{\alpha}} ds$$

$$\leq 2k \int_T^t (t-s)(s-T)^{\frac{1}{\alpha}} ds = 2kG(t,T),$$

and hence $\mathcal{F}y \in Y$. Thus, $\mathcal{F}$ maps Y into itself. Let $\{y_n\}$ be a sequence of functions in Y converging to $y \in Y$ in the metric topology of $C[T_*, \infty)$. Then, by using Lebesgue's dominated convergence theorem, we can prove that the sequence $\{\mathcal{F}y_n(t)\}$ converges to $\mathcal{F}y(t)$ as $n \to \infty$ uniformly on every compact interval of $[T_*, \infty)$, implying that $\mathcal{F}y_n \to \mathcal{F}y$ as $n \to \infty$ in $C[T_*, \infty)$. Hence $\mathcal{F}$ is a continuous mapping.
For any $y \in Y$ we have

$$(\mathcal{F}y(t))' = \int_T^t \left[\int_T^s \left(k^\alpha + \int_r^\infty q(\sigma)(y(g(\sigma)))^\beta d\sigma \right) dr \right]^{\frac{1}{\alpha}} ds, \quad t \geq T,$$

which implies that

$$0 \leq (\mathcal{F}(t))' \leq 2k \int_T^t (s-T)^{\frac{1}{\alpha}} ds = \frac{2k\alpha}{\alpha+1}(t-T)^{1+\frac{1}{\alpha}}, \quad t \geq T.$$

From this inequality, together with the fact that $\mathcal{F}y \in Y$, we conclude that the set $\mathcal{F}(Y)$ is relatively compact in the topology of $C[T_*, \infty)$. Therefore, by the Schauder-Tychonoff fixed point theorem, there exists a fixed element $y \in Y$ of $\mathcal{F}$, i.e., $y = \mathcal{F}y$, which satisfies the integral equation

$$y(t) = \int_T^t (t-s) \left[\int_T^s \left(k^\alpha + \int_r^\infty q(\sigma)(y(g(\sigma)))^\beta d\sigma \right) dr \right]^{\frac{1}{\alpha}} ds, \quad t \geq T. \tag{1.15}$$

This is a special case of (1.3) with $k_0 = k_1 = k_2 = 0$ and $\omega_3 = k^\alpha$. Differentiation of (1.15) shows that $y(t)$ is a positive solution of (A) on $[T, \infty)$. Since $\lim_{t\to\infty}((y''(t))^\alpha)' = k^\alpha > 0$, $y(t)$ is a desired solution of type I_1. This completes the proof.

D) *Nonoscillation criteria* (*sufficient conditions*). Let us now turn our attention to positive solutions of types I_2 and II_2 of (A). We are content with sufficient conditions for the existence of these two types of positive solutions of "intermidiate"growth. We observe that this kind of problem has not been dealt with even for ordinary differential equations without deviating arguments of the form (B); see Wu [6].

THEOREM 1.5. *The equation* (A) *has a positive solution of type* I_2 *if*

$$\int_0^\infty (g(t))^{(2+\frac{1}{\alpha})\beta} q(t)dt < \infty \tag{1.16}$$

and

$$\int_0^\infty t(g(t))^{2\beta} q(t)dt = \infty. \tag{1.17}$$

THEOREM 1.6. *The equation* (A) *has a positive solution of type* II_2 *if*

$$\int_0^\infty \left[\int_t^\infty (s-t)(g(s))^\beta q(s)ds\right]^{\frac{1}{\alpha}} dt < \infty \tag{1.18}$$

and

$$\int_0^\infty t\left[\int_t^\infty (s-t)q(s)ds\right]^{\frac{1}{\alpha}} dt = \infty. \tag{1.19}$$

2. Oscillation theorems

A) Our aim in this section is to establish criteria (preferably sharp) for the oscillation of all solutions of the equation (A). We are essentially based on some of the oscillation results of Wu [6], which are collected as Theorem W below, for the associated ordinary differential equation (B).

THEOREM W. (i) *Let* $\alpha \geq 1 > \beta$. *All solutions of* (B) *are oscillatory if and only if*

$$\int_0^\infty t^{(2+\frac{1}{\alpha})\beta} q(t)dt = \infty. \tag{2.1}$$

(ii) *Let* $\alpha \leq 1 < \beta$. *All solutions of* (B) *are oscillatory if and only if*

$$\int_0^\infty tq(t)dt = \infty \tag{2.2}$$

or

$$\int_0^\infty tq(t)dt < \infty \quad \text{and} \quad \int_0^\infty t\left[\int_t^\infty (s-t)q(s)ds\right]^{\frac{1}{\alpha}} dt = \infty. \tag{2.3}$$

B) *Comparison theorems.* Our idea is to deduce oscillation criteria for (A) from Theorem W by means of the following two lemmas (comparison theorems) which relate the oscillation (and nonoscillation) of the equation

$$(|u''(t)|^\alpha \operatorname{sgn} u''(t))'' + F(t, u(h(t))) = 0 \tag{2.4}$$

to that of the equations

$$(|v''(t)|^\alpha \operatorname{sgn} v''(t))'' + G(t, v(k(t))) = 0 \tag{2.5}$$

and

$$(|w''(t)|^\alpha \operatorname{sgn} w''(t))'' + \frac{l'(t)}{h'(h^{-1}(l(t)))} F(h^{-1}(l(t)), w(l(t))) = 0. \tag{2.6}$$

With regard to (2.4)–(2.6) it is assumed that $\alpha > 0$ is a constants, that h, k, l are continuously differentiable functions on $[0,\infty)$ such that

$$h'(t) > 0,\quad k'(t) > 0,\quad l'(t) > 0,\quad \lim_{t\to\infty} h(t) = \lim_{t\to\infty} k(t) = \lim_{t\to\infty} l(t) = \infty,$$

and that F, G are continuous functions on $[0,\infty)\times\mathbb{R}$ such that $uF(t,u) \geq 0$, $uG(t,u) \geq 0$ and $F(t,u)$, $G(t,u)$ are nondecreasing in u for any fixed $t \geq 0$. Naturally, h^{-1} denotes the inverse function of h.

LEMMA 2.1. *Suppose that*

$$h(t) \geq k(t), \quad t \geq 0 \tag{2.7}$$

$$F(t,x)\operatorname{sgn} x \geq G(t,x)\operatorname{sgn} x, \quad (t,x) \in [0,\infty)\times\mathbb{R}. \tag{2.8}$$

If all the solution of (2.5) *are oscillatory, then so are all the solutions of* (2.4).

LEMMA 2.2. *Suppose that* $l(t) \geq h(t)$ *for* $t \geq 0$. *If all the solution of* (2.6) *are oscillatory, then so are all the solutions of* (2.4).

These lemmas can be regarded as generalizations of the main comparison principles developed in the papers [2,4] to differential equations involving higher order nonlinear differential operators. To prove these lemmas we need a result which describes the equivalence of nonoscillation situation between (2.4) and the differential inequality

$$(|z''(t)|^{\alpha}\operatorname{sgn} z''(t))'' + F(t, z(h(t))) \leq 0. \tag{2.9}$$

LEMMA 2.3. *If there exists an eventually positive function satisfying* (2.9), *then* (2.4) *has an eventually positive solution.*

PROOF OF LEMMA 2.3. Let $z(t)$ be an eventually positive solution of (2.9). It is easy to see that $z(t)$ satisfies either

$$\text{I:}\qquad z'(t) > 0,\quad z''(t) > 0,\quad ((z''(t))^{\alpha *})' > 0,\quad t \geq T,$$

or

$$\text{II:}\qquad z'(t) > 0,\quad z''(t) < 0,\quad ((z''(t))^{\alpha *})' > 0,\quad t \geq T,$$

provided $T > 0$ is sufficiently large.

If $z(t)$ satisfies I, integrating (2.9) from t to ∞, we have

$$((z''(t))^{\alpha})' \geq \omega + \int_t^{\infty} F(s, z(h(s)))ds, \quad t \geq T, \tag{2.10}$$

where $\omega = \lim_{t\to\infty}((z''(t))^{\alpha})' \geq 0$. Further three integrations of (2.10) from T to t yield the inequality

$$z(t) \geq z(T) + \int_T^t (t-s)\left[\int_T^s \left(\omega + \int_r^{\infty} F(\sigma, z(h(\sigma)))d\sigma\right) dr\right]^{\frac{1}{\alpha}} ds, \quad t \geq T. \tag{2.11}$$

Let $T_* = \min\{T, \inf_{t\geq T} g(t)\}$. Put

$$U = \{y \in C[T_*,\infty) : \ 0 \leq u(t) \leq z(t),\ t \geq T_*\} \tag{2.12}$$

and define

$$\Phi u(t) = z(T) + \int_T^t (t-s)\left[\int_T^s \left(\omega + \int_r^\infty F(\sigma, u(h(\sigma)))d\sigma\right) dr\right]^{\frac{1}{\alpha}} ds, \quad t \geq T$$

(2.13)

$$\Phi u(t) = z(t), \qquad T_* \leq t \leq T.$$

Then, it is easily verified that Φ maps continuously U into a relatively compact set of U, and so there exists a function $u \in U$ such that $u = \Phi u$, which implies that

$$u(t) = z(T) + \int_T^t (t-s)\left[\int_T^s \left(\omega + \int_r^\infty F(\sigma, u(h(\sigma)))d\sigma\right) dr\right]^{\frac{1}{\alpha}} ds, \quad t \geq T. \tag{2.14}$$

This shows that $u(t)$ is a positive solution of the equation (2.4).

If $z(t)$ satisfies II, then (2.10) holds with $\omega = 0$, and integrating (2.10) from t to ∞, we find

$$-z''(t) \geq \left[\int_t^\infty (s-t)F(s, z(h(s)))ds\right]^{\frac{1}{\alpha}}, \quad t \geq T, \tag{2.15}$$

from which, integrating twice, first from t to ∞ and then from T to t, we obtain

$$z(t) \geq z(T) + \int_T^t \int_s^\infty \left[\int_r^\infty (\sigma - r)F(\sigma, z(h(\sigma)))d\sigma\right]^{\frac{1}{\alpha}} drds, \quad t \geq T. \tag{2.16}$$

Let $T_* = \min\{T, \inf_{t \geq T} g(t)\}$ and let U and Ψ be defined, respectively, by (2.12) and

$$\Psi u(t) = z(T) + \int_T^t \int_s^\infty \left[\int_r^\infty (\sigma - r)F(\sigma, u(h(\sigma)))d\sigma\right]^{\frac{1}{\alpha}} drds, \quad t \geq T,$$

(2.17)

$$\Psi u(t) = z(t), \qquad T_* \leq t \leq T.$$

The Schauder-Tychonoff fixed point theorem also applies to this case, and there exists a function $u \in U$ such that $u = \Psi u$, that is,

$$u(t) = z(T) + \int_T^t \int_s^\infty \left[\int_r^\infty (\sigma - r)F(\sigma, u(h(\sigma)))d\sigma\right]^{\frac{1}{\alpha}} drds, \quad t \geq T. \tag{2.18}$$

It follows that $u(t)$ is a positive solution of (2.4). This completes the proof of Lemma 2.3.

C) *Oscillation criteria.* We first give a sufficient condition for all solutions of (A) in the sub-half-linear case to be oscillatory.

THEOREM 2.1. *Let $\alpha \geq 1 > \beta$. Suppose that there exists a continuously differentiable function $h : [0,\infty) \to (0,\infty)$ such that $h'(t) > 0$, $\lim_{t\to\infty} h(t) = \infty$, and*

$$\min\{t, g(t)\} \geq h(t) \quad \textit{for} \;\; t \geq 0. \tag{2.19}$$

if

$$\int_0^\infty (h(t))^{(2+\frac{1}{\alpha})\beta} q(t)dt = \infty, \tag{2.20}$$

then all solutions of (A) *are oscillatory.*

THEOREM 2.2. *Let $\alpha \geq 1 > \beta$ and suppose that*

$$\limsup_{t\to\infty} \frac{g(t)}{t} < \infty. \tag{2.21}$$

Then, all solutions of (A) *all oscillatory if and only if*

$$\int_0^\infty (g(t))^{(2+\frac{1}{\alpha})\beta} q(t)dt = \infty. \tag{2.22}$$

An oscillation criterion for the equation (A) in the super-half-linear case is given in the following theorem.

THEOREM 2.3. *Let $\alpha \leq 1 < \beta$ and suppose that*

$$\liminf_{t\to\infty} \frac{g(t)}{t} > 0. \tag{2.23}$$

Then, all solution of (A) *are oscillatory if and only if either* (2.2) *or* (2.3) *holds.*

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